Exam 2 will be returned tomorrow
Closing *Tues*: 4.3
Closing *Thurs*: 4.4
Closing next *Tues*: 4.4-5
Closing next *Thurs*: 4.7 (last assignment)
I strongly suggest you finish 4.5 by the end of this week and so you can devote the last week to 4.7 and final studying.

4.3 Local Max/Min and 1st and 2nd derivative tests *(continued)*

Entry Task:
Find and classify the critical points for

$$y = 2 + 2x^2 - x^4$$

(*use the 1st deriv. test*)

$\boxed{\text{NOTE:}}$ INCREASING ON
$(-\infty, -1) \cup (0, 1)$
DECREASING ON
$(-1, 0) \cup (1, \infty)$
INTERVAL NOTATION

$y' = 4x - 4x^3$
$4x(1-x^2) = 4x(1-x)(1+x)$

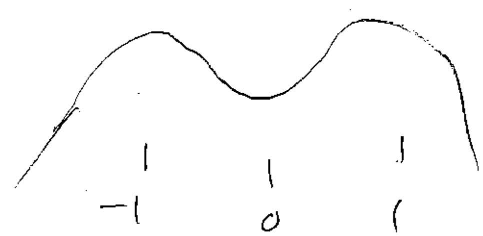
$\boxed{y' = 4x(1-x)(1+x)} \stackrel{?}{=} 0$

$x = 0$ or $x = 1$ or $x = -1$

y	↗		↘		↗		↘
y'	+	-1	−	0	+	1	−

$4x(1-x)(1+x)$: − + − | $4x(1-x)(1+x)$: − + + | $4x(1-x)(1+x)$: + + + | $4x(1-x)(1+x)$: + − +

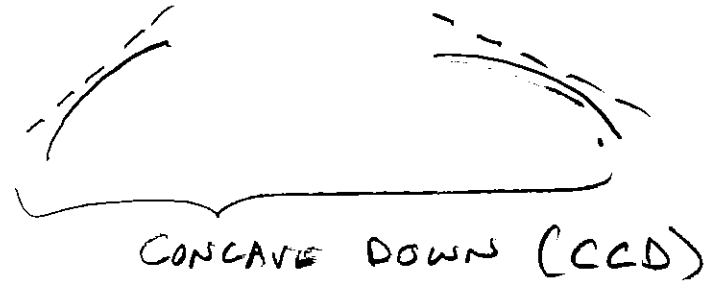
$x = -1$ LOCAL MAX
$x = 0$ LOCAL MIN
$x = 1$ LOCAL MAX



The 2nd Derivative

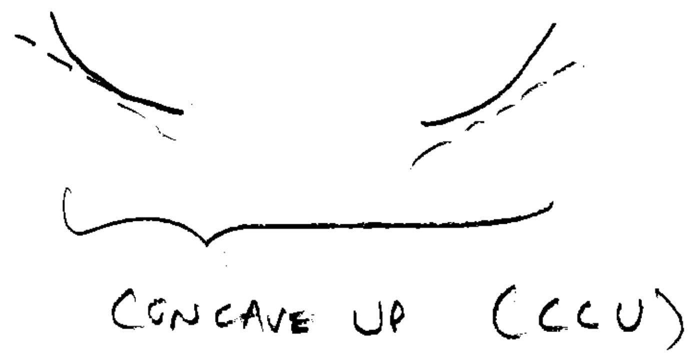
$$y'' = f''(x) = \frac{d}{dx}(f'(x))$$

= "rate of change of 1st deriv."



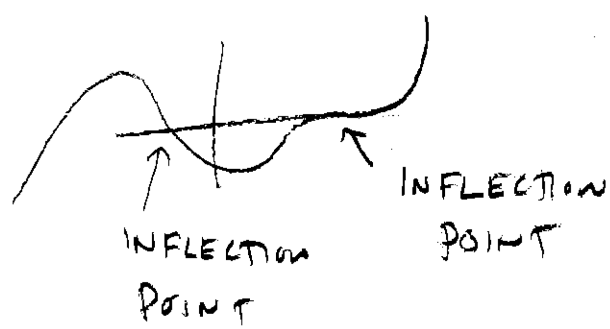
Terminology

If **f''(x) is positive**,
then the **slope of f(x) is *increasing***
and we say f(x) is **concave up**.



If **f''(x) is negative**,
then the **slope of f(x) is *decreasing***
and we say f(x) is **concave down**.

A point in the domain of the function at which the concavity changes is called an **inflection point**.



Summary:

$y = f(x)$	$y'' = f''(x)$
possible inflection	zero
concave up	positive
concave down	negative
possible inflection	does not exist

Example: Find all inflection points and indicate where the function is concave up and concave down for

$$y = x^4 - 2x^3$$

$y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x \overset{?}{=} 0$

$12x(x-1) = 0$

$\boxed{x = 0,\ x = 1}$

y	$\cup$		$\cap$		$\cup$
y''	$+$	0	$-$	1	$+$

$y'' = 12x(x-1)$ | $12x(x-1)$ | $12x(x-1)$

$-\ \ -$ | $+\ \ -$ | $+\ \ +$

$x = 0$ INFLECTION POINT

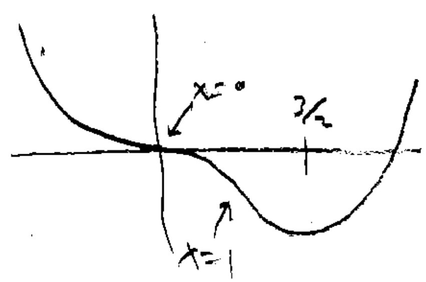
$x = 1$ INFLECTION POINT

NOTE: $4x^3 - 6x^2 \overset{?}{=} 0$

$2x^2(2x-3) = 0$

$x = 0$ or $x = 3/2$

y	$\searrow$		$\searrow$		$\nearrow$
y'	$-$	0	$-$	$3/2$	$+$



Clever Observation

(*Second Derivative Test*)

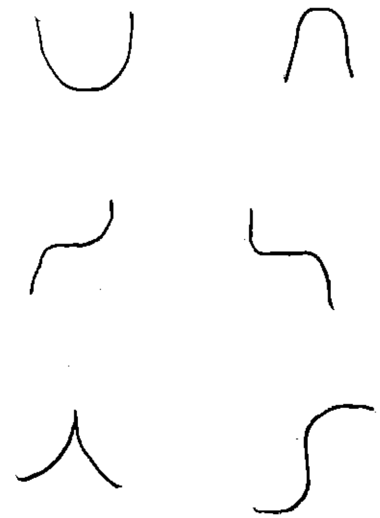
If $x = a$ is a critical number for $f(x)$

AND

1. if $f''(a)$ is positive (CCU),
   then a local min occurs at $x = a$.

2. if $f''(a)$ is negative (CCD),
   then a local max occurs at $x = a$.

3. if $f''(a) = 0$,
   then we say the 2nd deriv. test is
   inconclusive (need other method)



Example: Find and classify the critical numbers for

$$y = 2 + 2x^2 - x^4$$

(use the 2^{nd} deriv. test)

$$y' = 4x - 4x^3 \stackrel{?}{=} 0$$

$$4x(1 - x^2) = 0$$

$$x = 0, -1, 1$$

$$y'' = 4 - 12x^2$$

$x = -1 \Rightarrow y''(-1) = 4 - 12(-1)^2 = -8 < 0$ CCD $\Rightarrow$ LOCAL MAX

$x = 0 \Rightarrow y''(0) = 4 - 12(0)^2 = 4 > 0$ CCU $\Rightarrow$ LOCAL MIN

$x = 1 \Rightarrow y''(1) = 4 - 12(1)^2 = -8 < 0$ CCD $\Rightarrow$ LOCAL MAX

4.4 L'Hopital's Rule

First, recall as we discussed many, many, many times at the beginning of the term:

(Assuming f and g are cont. at *x*=*a*)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = ??$$

- If $g(a) \neq 0$, then done!

  Ans = $\frac{f(a)}{g(a)}$.

- If $g(a) = 0$ and $f(a) \neq 0$, then examine each side of $x = a$ (look at the signs)

  Ans = ∞, $-\infty$, or DNE.

- If $g(a) = 0$ and $f(a) = 0$, then use algebra to rewrite and 'cancel' the denominator.

L'Hopital's Rule (0/0 case)

Suppose $g(a) = 0$ and $f(a) = 0$ and f and g are differentiable at $x = a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Examples:

1. $$\lim_{x\to 4}\frac{16-x^2}{4-x} \overset{0/0}{=} \lim_{x\to 4}\frac{\cancel{(4-x)}(4+x)}{\cancel{(4-x)}} = \boxed{8}$$

Or

$$\lim_{x\to 4}\frac{16-x^2}{4-x} \overset{\boxed{H}}{=} \lim_{x\to 4}\frac{-2x}{-1} = \lim_{x\to 4} 2x = \boxed{8}$$

2. $$\lim_{x\to 0}\frac{\sin(x)}{x} = \boxed{1}$$ ← From 3.3

Or

$$\lim_{x\to 0}\frac{\sin(x)}{x} \overset{0/0}{\underset{\boxed{H}}{=}} \lim_{x\to 0}\frac{\cos(x)}{1} = \boxed{1}$$

3. $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} \overset{0/0}{=} \lim_{x\to 0}\frac{\cancel{1}+x\cancel{-1}}{x(\sqrt{1+x}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{1+x}+1} = \boxed{\frac{1}{2}}$$

Or

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} \overset{0/0}{\underset{H}{=}} \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \boxed{\frac{1}{2}}$$

Aside: Sketch of derivation

Assume $g(a) = 0$ and $f(a) = 0$

(These explanations are for the case when g'(a) is not zero).

Explanation 1 (def'n of derivative)

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

provided these limits exist we have:

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Explanation 2 (tangent line approx.):

The tangent lines for $f(x)$ and $g(x)$ at $x = a$ are

$$y = f'(a)(x-a) + 0$$

$$y = g'(a)(x-a) + 0$$

And we know these approximate the functions f(x) and g(x) better and better the closer x gets to a, so

Thus,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$